

13.3 Measurements on Curves in 3D

Today: Unit Tangent and Normal, Arc Length, and Curvature

Entry Task

Consider $\mathbf{r}(t) = \langle t, t^2, 5 \rangle$.

Find the unit tangent vector $\mathbf{T}(t)$.

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 0}$$

$$\boxed{\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}, 0 \right\rangle}$$

NOTE
 $\mathbf{T} \cdot \mathbf{T}' =$

$$= \frac{4t}{(1+4t^2)^2} + \frac{4t}{(1+4t^2)^2} = 0 \quad \checkmark$$

Thm: \mathbf{T} and \mathbf{T}' are always orthogonal.

Proof: Since $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$,
we differentiate both sides to get

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0.$$

So $2\mathbf{T} \cdot \mathbf{T}' = 0$.

Thus, $\mathbf{T} \cdot \mathbf{T}' = 0$. (QED)

ASIDE

$$\frac{d}{dt} \left((1+4t^2)^{-1/2} \right) = -\frac{1}{2} (1+4t^2)^{-3/2} \cdot 8t = \frac{-4t}{(1+4t^2)^{3/2}}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{2t}{(1+4t^2)^{1/2}} \right) &= \left(\frac{\sqrt{1+4t^2} \cdot 2 - 2t \cdot \frac{8t}{2\sqrt{1+4t^2}}}{1+4t^2} \right) \frac{\sqrt{1+4t^2}}{\sqrt{1+4t^2}} \\ &= \frac{(1+4t^2) \cdot 2 - 8t^2}{(1+4t^2)^{3/2}} \\ &= \frac{2}{(1+4t^2)^{3/2}} \end{aligned}$$

↑
SIMPLY

$$\Rightarrow \boxed{\mathbf{T}'(t) = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}, 0 \right\rangle}$$

13.3 Measurements on Curves in 3D

Distance Traveled on a Curve

The dist. traveled along a curve from $t = a$ to $t = b$ is

$$\int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Note: 2D is same without the $z'(t)$.

We derived this in Math 125.

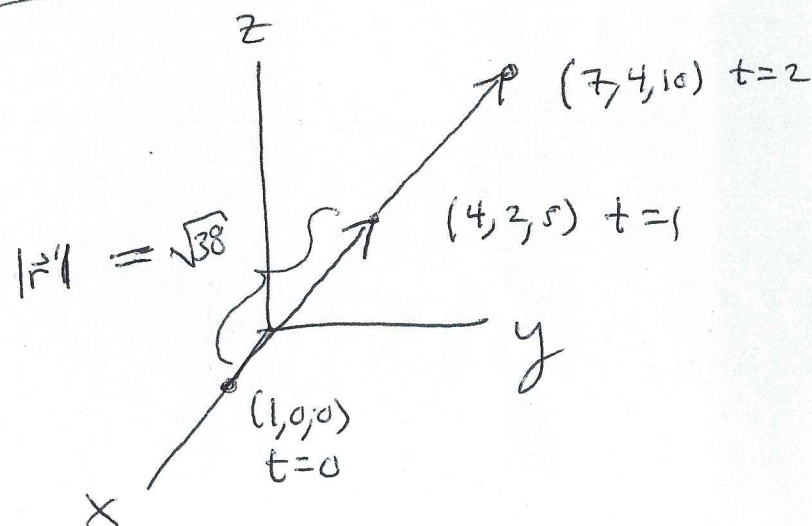
Examples:

- Find the length of the curve
 $\mathbf{r}(t) = \langle 1 + 3t, 2t, 5t \rangle$
from $t = 0$ to $t = 2$.

$$\begin{aligned}x'(t) &= 3 \\y'(t) &= 2 \\z'(t) &= 5\end{aligned}$$

$$\begin{aligned}\text{DISTANCE} &= \int_0^2 \sqrt{(3)^2 + (2)^2 + (5)^2} dt \\&= \int_0^2 \sqrt{9 + 4 + 25} dt \\&= \int_0^2 \sqrt{38} dt \\&= \sqrt{38} t \Big|_0^2 \\&= 2\sqrt{38}\end{aligned}$$

NOTE: LINEAR MOTION



13.3 Measurements on Curves in 3D

Goal: distance/arc length,
unit tangent, unit normal, curvature.

Entry Task: Find the length of the curve

$$\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 2 \ln(\cos(t)) \rangle$$

$0 \text{ to } \frac{\pi}{3}$

$$x'(t) = -2 \sin(2t)$$

$$y'(t) = 2 \cos(2t)$$

$$z'(t) = 2 \frac{1}{\cos(t)} \sin(t) = 2 \tan(t)$$

Distance Traveled on a Curve

The dist. traveled along a curve from
 $t = a$ to $t = b$ is

$$\int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Note: 2D is same without the $z'(t)$. We derived this in Math 125.

If the curve is "traversed once" we call this **arc length**.

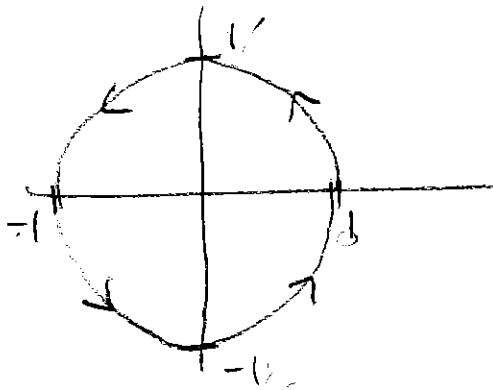
$$\begin{aligned} &= \int_0^{\pi/3} \sqrt{4 \sin^2(2t) + 4 \cos^2(2t) + 4 \tan^2(t)} dt \\ &= \int_0^{\pi/3} \sqrt{4 + 4 \tan^2(t)} dt \\ &= 2 \int_0^{\pi/3} \sqrt{1 + \tan^2(t)} dt \\ &= 2 \int_0^{\pi/3} \sqrt{\sec^2(t)} dt \\ &= 2 \int_0^{\pi/3} \sec(t) dt \\ &= 2 \ln |\sec(t) + \tan(t)| \Big|_0^{\pi/3} \end{aligned}$$

$$= 2 \ln \left| \underbrace{\sec(\frac{\pi}{3})}_2 + \underbrace{\tan(\frac{\pi}{3})}_{\sqrt{3}} \right| - 2 \ln \left| \underbrace{\sec(0)}_1 + \underbrace{\tan(0)}_0 \right|$$

$$= \boxed{2 \ln(2 + \sqrt{3})} \approx 2.63$$

Example: $x = \cos(t)$, $y = \sin(t)$

- (a) Find the distance traveled by this object from $t = 0$ to $t = 6\pi$.
- (b) Find the arc length of the path over which this object is traveling.



$$\int_0^{6\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2} dt$$

$$\int_0^{6\pi} \sqrt{1} dt$$

$$\int_0^{6\pi} 1 dt = t \Big|_0^{6\pi}$$

$$= \boxed{6\pi} = \text{DISTANCE TRAVELED}$$

TRAVELLED ONCE From $t=0$ to $t=2\pi$

$$\int_0^{2\pi} \sqrt{1} dt = \boxed{2\pi} = \text{ARC LENGTH}$$

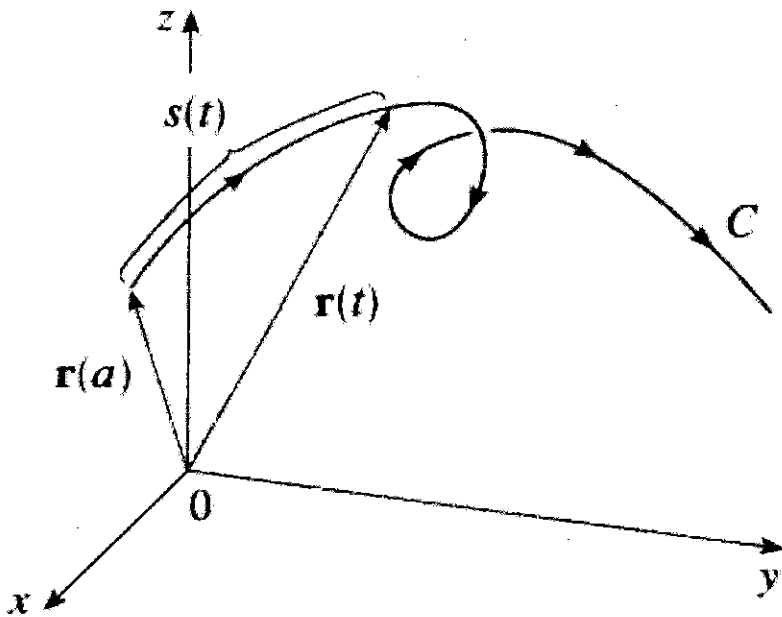
Arc Length Function

The distance from 0 to t is called the *arc length function*

$$s(t) = \int_0^t |\vec{r}'(u)| du = \text{distance}$$

Note:

$$\frac{ds}{dt} = |\vec{r}'(t)| = \text{speed}$$



Example: $x = 3 + 2t$, $y = 4 - 5t$

- (a) Find the arc length (from 0 to t).
(b) **Reparameterize** in terms of $s(t)$.

$$\begin{aligned} s &= \int_0^t \sqrt{(2)^2 + (-5)^2} du \\ &= \int_0^t \sqrt{29} du \\ &= \sqrt{29} u \Big|_0^t \end{aligned}$$

$$\Rightarrow \boxed{s = \sqrt{29} t}$$

↑ DISTANCE ↑ TIME

← SPEED

THUS, $t = \frac{1}{\sqrt{29}} s$

$$\boxed{\begin{aligned} x &= 3 + 2t = 3 + \frac{2}{\sqrt{29}} s \\ y &= 4 - 5t = 4 - \frac{5}{\sqrt{29}} s \end{aligned}}$$

IN TERMS OF TIME IN TERMS OF DIST.

Unit Tangent & Principal Unit Normal

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \text{unit tangent}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \text{principal unit normal}$$

AT A "LOW"
POINT

Example:

$$\vec{r}(t) = \langle 2 \sin(3t), t, 2 \cos(3t) \rangle$$

Find $\vec{T}(\pi)$ and $\vec{N}(\pi)$

$$\vec{r}'(t) = \langle 6 \cos(3t), 1, -6 \sin(3t) \rangle$$

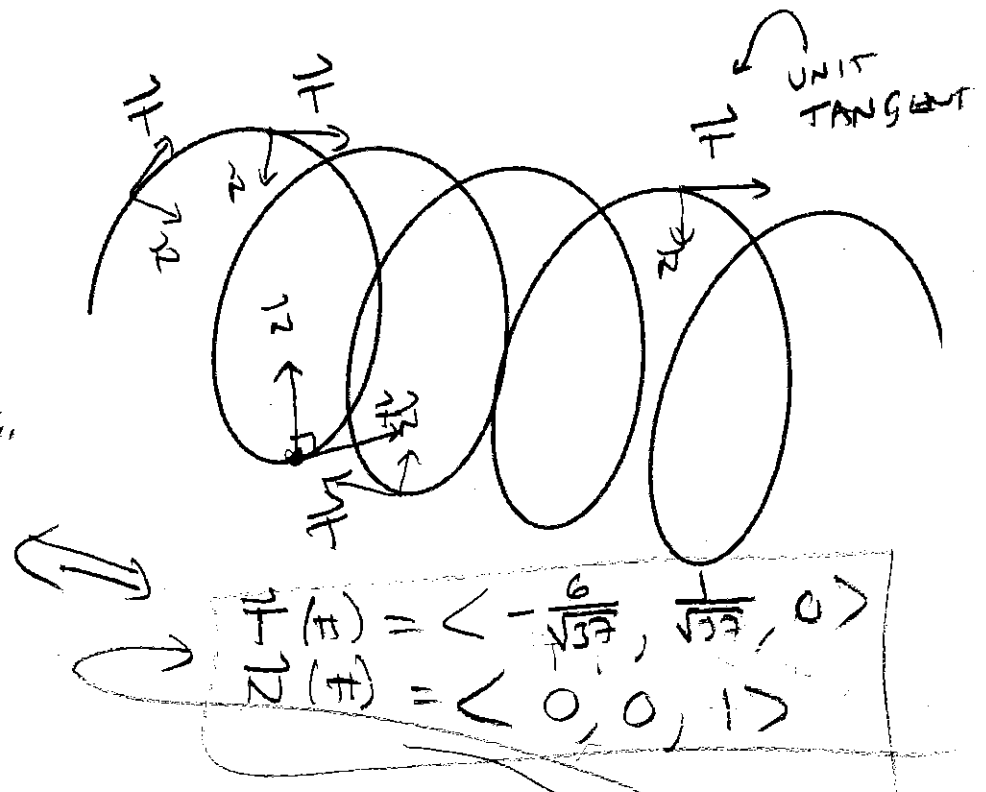
$$|\vec{r}'(t)| = \sqrt{36 \cos^2(3t) + 1 + 36 \sin^2(3t)} = \sqrt{37}$$

$$\vec{T}(t) = \frac{1}{\sqrt{37}} \vec{r}'(t) = \left\langle \frac{6}{\sqrt{37}} \cos(3t), \frac{1}{\sqrt{37}}, \frac{-6}{\sqrt{37}} \sin(3t) \right\rangle$$

$$\vec{T}'(t) = \left\langle -\frac{18}{\sqrt{37}} \sin(3t), 0, -\frac{18}{\sqrt{37}} \cos(3t) \right\rangle$$

$$|\vec{T}'(t)| = \sqrt{\left(\frac{18}{\sqrt{37}}\right)^2 (\sin^2(3t) + \cos^2(3t))} = \frac{18}{\sqrt{37}}$$

$$\vec{N}(t) = \frac{1}{18/\sqrt{37}} \vec{T}'(t) = \langle -\sin(3t), 0, -\cos(3t) \rangle$$



$$\vec{T}(\pi) = \left\langle -\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0 \right\rangle$$

$$\vec{N}(\pi) = \langle 0, 0, 1 \rangle$$

Why does this work?

\mathbf{T} and \mathbf{T}' are always orthogonal.

Proof:

Since $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$,

we can differentiate both sides to get

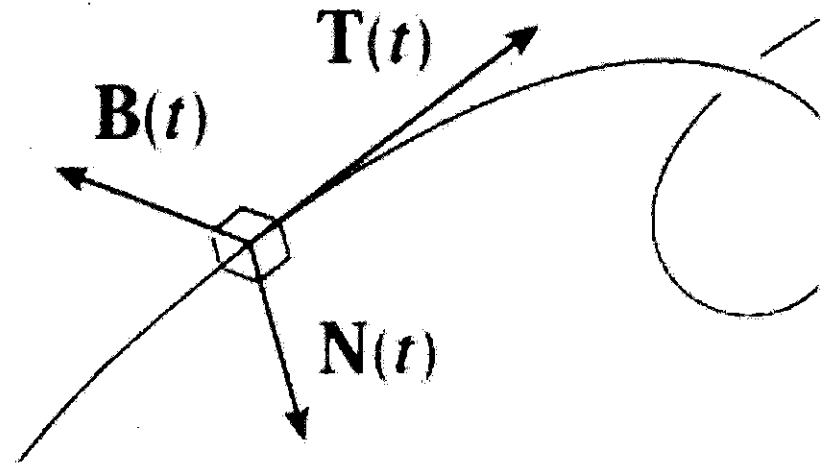
$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0.$$

So $2\mathbf{T} \cdot \mathbf{T}' = 0$.

Thus, $\mathbf{T} \cdot \mathbf{T}' = 0$. (QED)

Some TNB-Frame Facts:

- $\vec{T}(t)$ and $\vec{N}(t)$ point in the tangent and *inward* directions, respectively. Together they give a good approximation of the “plane of motion”. This “plane of motion” that goes through a point on the curve and is parallel to $\vec{T}(t)$ and $\vec{N}(t)$ is called the *osculating (kissing) plane*.
- $\vec{T}(t)$, $\vec{N}(t)$, $\vec{r}'(t)$, and $\vec{r}''(t)$ are ALL parallel to the osculating plane. We also define
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \text{binormal}$$
which is orthogonal to all of $\vec{T}(t)$, $\vec{N}(t)$, $\vec{r}'(t)$, and $\vec{r}''(t)$.



Curvature

The **curvature** at a point, K , is a measure of how quickly a curve is changing direction at that point.

That is, we define

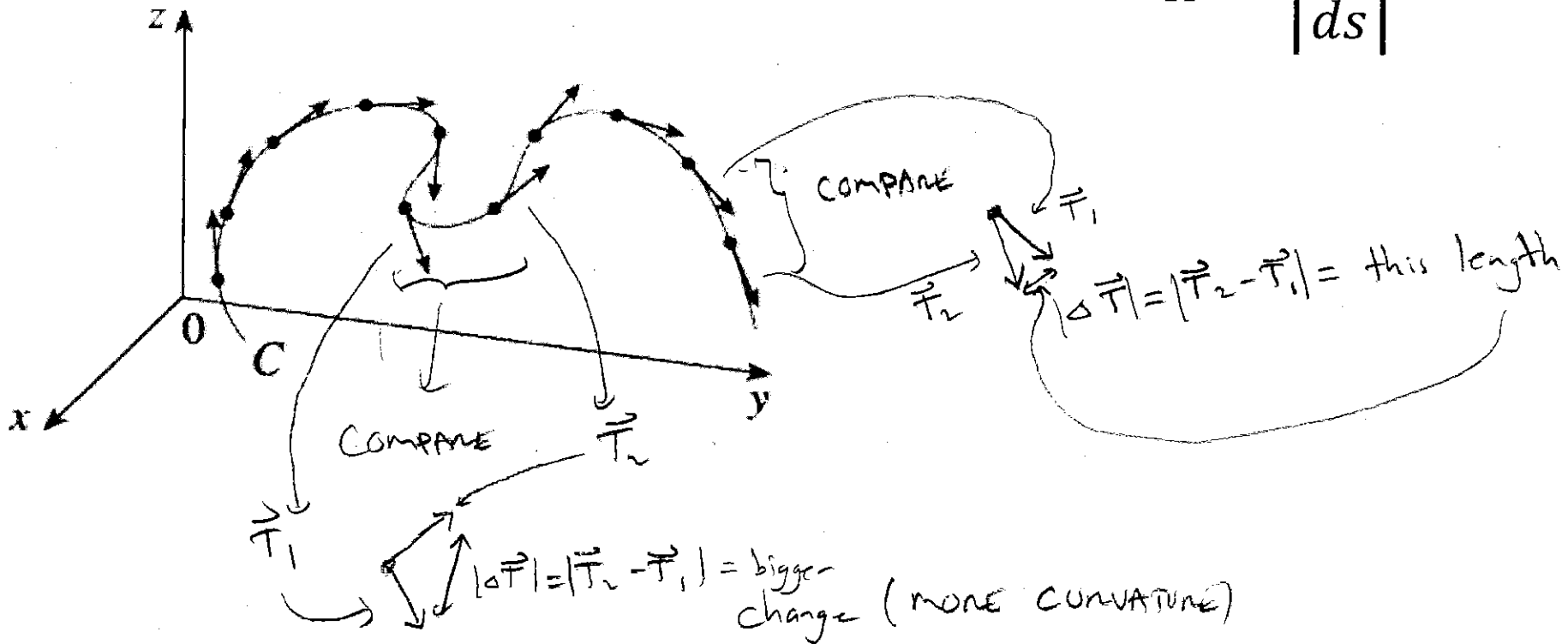
$$K = \frac{\text{change in direction}}{\text{change in distance}}$$

Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left| \frac{\vec{T}_2 - \vec{T}_1}{\text{"one inch"}} \right| = \left| \frac{\Delta \vec{T}}{\Delta s} \right|$$

So we define:

$$K = \left| \frac{d\vec{T}}{ds} \right|$$



Computation

$$K = \left| \frac{d\vec{T}}{ds} \right|$$

is not easy to compute directly, so we derive some *shortcuts*

1st shortcut:

From math 124

$$K(t) = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

2nd shortcut

$$K(t) = \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

TAKES SOME
WORK TO DERIVE.

(SEE LATER IN THESE
NOTES)

Example: Find the curvature function for $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$.

$$\mathbf{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$$

$$\mathbf{r}''(t) = \langle 0, -4\cos(2t), 4\sin(2t) \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}$$

$$\text{so } |\mathbf{r}'(t)| = \sqrt{5}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -8, -4\sin(2t), -4\cos(2t) \rangle$$

$$\text{So } |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{64 + 16} = \sqrt{80}$$

$$\frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3} = \frac{\sqrt{80}}{\sqrt{5}^3} = \sqrt{\frac{80}{125}} = 0.8$$

This curve has constant curvature!

Proof of shortcut:

Theorem: $\frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

Proof:

Since $T(t) = \frac{r'(t)}{|r'(t)|}$, we have

$$r'(t) = |r'(t)|T(t).$$

Differentiating this gives (prod. rule):

$$r''(t) = |r'(t)|'T(t) + |r'(t)|T'(t).$$

Take cross-prod. of both sides with \vec{T} :

$$T \times r'' = |r'|' (T \times T) + |r'| (T \times T').$$

Since $T \times T = \langle 0, 0, 0 \rangle$ (why?)

and $T = \frac{r'}{|r'|}$, we have

$$\frac{r' \times r''}{|r'|} = |r'| (T \times T').$$

Taking the magnitude gives (why?)

$$\frac{|r' \times r''|}{|r'|} = |r'| |T \times T'| = |r'| |T| |T'| \sin\left(\frac{\pi}{2}\right),$$

Since $|T| = 1$, we have

$$|T'| = \frac{|r' \times r''|}{|r'|^2}$$

Therefore

$$K = \left| \frac{dT}{ds} \right| = \frac{|T'(t)|}{|r'(t)|} = \frac{|r' \times r''|}{|r'|^3}.$$

Note: To find curvature for a 2D function, $y = f(x)$, we can form a 3D vector function as follows

$$\mathbf{r}(x) = \langle x, f(x), 0 \rangle$$

so $\mathbf{r}'(x) = \langle 1, f'(x), 0 \rangle$ and

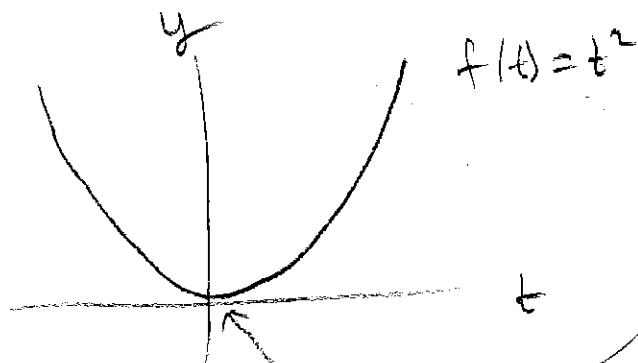
$$\mathbf{r}''(x) = \langle 0, f''(x), 0 \rangle$$

$$|\mathbf{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\mathbf{r}' \times \mathbf{r}'' = \langle 0, 0, f''(x) \rangle$$

Thus,

$$K(x) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$



Example: $f(t) = t^2$

Consider $x = t, y = t^2, z = 0$.

At what point (x, y, z) is the curvature maximum?

$$f'(t) = 2t$$

$$f''(t) = 2$$

$$K = \frac{2}{(1 + (2t)^2)^{3/2}}$$

$$k(t) = \frac{2}{(1 + 4t^2)^{3/2}} = 2(1 + 4t^2)^{-3/2}$$

$$K'(t) = -3(1 + 4t^2)^{-5/2} \cdot 8t = 0 \quad \boxed{t=0}$$

MAXIMUM CURVATURE!

Summary of 3D Curve Measurement Tools:

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}'(t)$ = a tangent vector

$$s(t) = \int_0^t |\vec{r}'(t)| dt$$

$$K = \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \text{unit tangent}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \text{principal unit normal}$$